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Random Incidence Matrices: Moments of the Spectral Density

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We study numerically and analytically the spectrum of incidence matrices of random labeled graphs on N vertices: any pair of vertices is connected by an edge with probability p. We give two algorithms to compute the moments of the eigenvalue distribution as explicit polynomials in N and p. For large N and fixed p the spectrum contains a large eigenvalue at Np and a semicircle of "small" eigenvalues. For large N and fixed average connectivity pN (dilute or sparse random matrices limit) we show that the spectrum always contains a discrete component. An anomaly in the spectrum near eigenvalue 0 for connectivity close to e is observed. We develop recursion relations to compute the moments as explicit polynomials in pN. Their growth is slow enough so that they determine the spectrum. The extension of our methods to the Laplacian matrix is given in Appendix.

KEY WORDS: Random graphs; random matrices; sparse matrices; incidence matrices spectrum; moments.

1. INTRODUCTION

The spectral properties of the incidence matrix of random graphs have motivated a large number of studies over the last decades. The same problem is described under rather different names, depending upon the aspects that are under focus and the method of attack.

The interest in this problem has several roots in physics. The replacement of complicated hamiltonians by large random matrices has proved very efficient in the analysis of the spectral properties (culminating in level spacing distributions) of large nuclei since the pioneering works of Wigner and Dyson. For many properties, the details of the probabilistic laws

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governing the distribution of matrix elements are irrelevant, and there is a very powerful notion of universality. Further motivation for considering precisely random incidence matrices comes from several systems in condensed matter physics, a good example being conductors with impurities. The pure system is modeled by a lattice, and electrons can move along bonds. Impurities break bonds. So the hamiltonian can be approximated by the incidence matrix of the lattice with random bonds removed. If one considers several large samples differing by the impurity concentration c, the following properties are observed. When c is large, only small islands of metallic atoms exist. If c decreases to reach a certain threshold, a large island of metallic atoms invades the system. This is classical percolation. However it is generally believed that the system remains insulating (the wave function of the electrons are all localized) until another threshold in the impurity concentration. Then some delocalized states appear and the sample is a conductor. This is called quantum percolation, a kind of Anderson (de)localization. The analytic study of this problem on a 3d lattice with random bonds removed is very difficult, and this motivated people to look at the much simpler problem of a random graph. This forgets about the spatial structure and is a kind of mean field approximation.

In the random graph model, the lattice is replaced by the complete graph on N points: any two points are connected by a bond (by an edge in the language of graph theory). Then, bonds are randomly removed, leading to a random graph where only a fraction p of the initial bonds remains. In the simplest case, bonds are removed with probability $1-p$ independently of each other. The model can be made more complicated by choosing randomly a sign for each bond present in the random graph. This allows interferences if the probability amplitude for an electron moving on the random graph is the product of the signs of the visited bonds.

The topology of large random graphs was investigated about four decades ago by Erdös and Renyi[9] in a remarkable series of papers. The idea is to let p vary with N. There are quite a few different regimes. The most relevant for further physical investigations are:

- The edge-probability p remains fixed as N goes to infinity.
- The average connectivity $\alpha = pN$ remains fixed as N goes to infinity.

In the first case, the infinite random graph is connected and in a precise sense two infinite random graphs of given p are isomorphic with probability 1. The second case exhibits a percolation transition at $\alpha = 1$. For small α all connected components are finite, and only trees contribute to the extensive quantities. But for $\alpha > 1$, a finite fraction of the points lies in a single connected component.

As explained above, the spectral properties of the (signed) incidence matrix of the random graph, a symmetric matrix with $0, (\pm)1$ matrix elements have a great physical interest. Quite often, authors concentrate on the case when the distribution of random signs is symmetric. Numerical simulations and analytic (mostly supersymmetric) methods have given a great amount of information.

For large N, fixed p and symmetric random signs, it is known that the spectral distribution is a semicircle[21] and that the level correlations are those of the Gaussian orthogonal ensemble (GOE), one of the four standard universality classes governing random spectra.[20]

The finite connectivity limit has also attracted a lot of attention, under the names of dilute or sparse random matrices. It has been argued[11, 12, 20] that there is some value $\alpha_q > 1$ for which delocalized eigenstates appear. So this simple model is believed to exhibit a quantum percolation transition.

Our aim in this paper is to use combinatorial methods to compute explicitly moments of the spectral distribution for given N and p, and in the finite connectivity limit.

Section 2 gives a formal definition of the model and recalls some of its topological properties. Section 3 gives the enumerative algorithm for moments as polynomials in N and p. Section 4 concentrates on the fixed p large N limit, first numerically (spectrum and level spacing) and then analytically. Section 5 deals with the finite connectivity large N limit, starting with numerical computations. In particular, we observe a quantitative change in the spectrum near the eigenvalue $\lambda = 0$ for $\alpha \simeq 2.7$. We give qualitative arguments for the presence, location and size of delta peaks in the spectrum. Then we derive a formal expression for the moment generating function and give a recursion relation for the moments, that we use to control their growth. In an Appendix, we show how our algorithms can be extended when the incidence matrix is replaced by the Laplacian matrix of a random graph.

2. THE MODEL

For $N = 1, 2, \ldots$, we define the sample space Ω_N as follows: the elements of Ω_N are the symmetric $N \times N$ matrices $M = (M_{ij})_{i,\, j \in [1,\, N]}$, with $M_{i,\, i} = 0$ on the diagonal and $M_{i,\, j} = 0$ or 1 for $i \neq j$. So Ω_N is a discrete space consisting of $2^{N(N-1)/2}$ points. We observe that Ω_N is in one to one correspondence with the set of labeled *simple* graphs on N vertices: to $M \in \Omega_N$, we associate the graph with vertex set $\{1, \ldots, N\}$ and edge set $E(M) = \{\{i, j\} \mid M_{ij} = 1\}$. Then $|E(M)| = \frac{1}{2} \sum_{i,\, j} M_{ij} = \sum_{i<j} M_{ij}$ is just the number of edges of the graph associated to M. The word *simple* above refers to the

fact that the graphs we consider do not contain multiple edges or edges with only one vertex. In the sequel, graph always means simple graph, and we talk indiscriminately of matrices or associated graphs.

For any $p \in [0, 1]$ we turn Ω_N into a probability space: the weight of $M \in \Omega_N$ is $P(M) = p^{E(M)}(1-p)^{N(N-1)/2 - E(M)}$. To rephrase this formal definition, the entries of M above the main diagonal are independent random variables with the same Bernoulli (binomial) distribution: for $i < j$, $M_{ij} = 1$ (or equivalently the vertices i and j are connected by an edge) with probability p and $M_{ij} = 0$ with probability $1 - p$.

The quantity $\alpha \equiv pN$ appears as the average connectivity,[2] i.e., the average number of vertices j connected by an edge $\{i, j\}$ to a given vertex i. Remark that this connectivity fluctuates, in contrast to regular graphs[16] where the connectivity is fixed for all vertices.

We can define a variant of this model by introducing a random sign, with a parameter $a \in [0, 1]$: for $i < j$, $M_{ij} = +1$ with probability ap, $M_{ij} = -1$ with probability $(1-a)\,p$ and $M_{ij} = 0$ with probability $1 - p$. The even model $a = 1/2$, which gives $\langle M_{ij} \rangle = 0$, has been studied by some authors.[11, 12, 20–22] If a random graph contains no loops (i.e., closed circuits), the parameter a is not relevant because all the negative signs can be changed in positive ones by a simple change of basis. More generally, it is true if the graph has no "frustrated" loops, i.e., no loops with odd number of negative edges. We will see later that the random spectrum is not sensitive to a in the large N limit with fixed α. So, without explicit indications, we will speak about the signless model defined previously, which has $a = 1$.

If X is any random variable on Ω_N, we use the notation $\bar{X}$ for the expectation value (or average) of X. In the next sections, we shall be interested in the asymptotic behavior of the moments of the spectral density $\overline{\mathrm{Tr}\, M^k}$ when $N \to \infty$, first with p fixed and then with $\alpha = Np$ fixed. But first, we recall a few fundamental facts on the topology of random graphs. The basic reference is ref. 9; for a textbook presentation and more references, see, e.g., ref. 6.

It is well established that this model has a *percolation* transition at $\alpha = 1$. In the regime $\alpha < 1$, with probability 1 in the large N limit, all the connected components of the random graph are finite: moreover they are mostly trees, there is only a finite number of one loop components and no

[2] The terminology *average connectivity* seems to be well established in the physics literature, and we stick to it. It would however be more consistent with general graph theory to call it the *average degree* (of vertices).

other connected components. For a given tree T on n vertices, the average number of connected components isomorphic to T is

$$\frac{N}{|\mathrm{Aut}(T)|}\alpha^{n-1}e^{-n\alpha}+o(N) \tag{1}$$

where $|\mathrm{Aut}(G)|$ for a given graph G is defined as the order of its automorphism group, formed by the permutations of the vertices that leave its incidence matrix invariant.

In the regime $\alpha>1$, Eq. (1) remains valid, but one "giant" connected component (equivalent of the percolation cluster for regular lattices) appears, with a finite fraction of the N vertices and many loops. This fraction is an increasing function of α, covering $[0,1]$ when α runs from 1 to ∞. In the limit N large with p fixed, the random graphs is made only with one component.

The percolation transition at $\alpha=1$ is of second order. As usual, critical exponents can be defined: by example, the biggest component has a size of order $N^{2/3}$.

Moreover this model exhibits[20] an Anderson *localization* transition, also called *quantum percolation* transition, at a value $\alpha_q>1$. In the phase $\alpha<\alpha_q$, all eigenvectors of the random incidence matrix are localized. On the other hand, for $\alpha>\alpha_q$, the eigenvectors for which the absolute value of the energy is below a threshold $E(\alpha)$ becomes extended.

By studying nearest level spacing between eigenvalues, we expect an exponential distribution of spacing in the localized phase (because the spatial covering between different eigenvectors vanishes), and a GOE distribution[19] in the delocalized phase. With this kind of criterion, the location transition has been numerically estimated[11, 12] at $\alpha_q\approx 1.4$. By considering quantum percolation on a randomly diluted Cayley (or Bethe) tree,[11, 12, 15] it has been conjectured that α_q is given by $\alpha_q\log\alpha_q=1/2$, (leading to $\alpha_q\approx 1.4215299$). We argued[3] that this value is exact for random incidence matrices, but that loops have nevertheless some influence on the localization properties.

So the percolation transition has a drastic effect on the topology of the random graph and the localization transition changes the behavior of eigenvectors of its incidence matrix, but as we shall see later, the transitions have no obvious impact on the moments of the spectral distribution of a random matrix. In fact, the situation may look paradoxical: many relevant quantities for the spectrum of random incidence matrices (for instance the moments) can be computed by looking at local structures on the random graph. For fixed α and large N, such structures are trees with probability 1. Hence loops that appear for $\alpha>1$ seem to play no role. However, the

presence of loops is crucial to ensure that the statistics of finite structures varies smoothly with α. For example, if, instead of random graphs, one considers random forests (union of trees),[4] one finds that for $\alpha < 1$, the thermodynamical properties are exactly equal to the ones for the random graph model, but the transition at $\alpha = 1$ (when an infinite tree appears) changes the distribution of local structures, and for instance the moments of the spectral distribution are not analytic at $\alpha = 1$ for random forests.

3. COMPUTATION OF MOMENTS

In this section, we derive Eq. (2), valid for any N and edge parameter p, which allows, for a given k, to compute directly $\overline{\operatorname{Tr} M^k}$, (i.e., N times the kth moment of the density of states), when M is a random incidence matrix in Ω_N. A sum rule for $p = 1$ is given. Then the algorithm is adapted for the variant of the model with random signs. Finally we give a compact formula, Eq. (6), for the generating function of moments.

3.1. Direct Computation

For any random incidence matrix M in Ω_N, $\operatorname{Tr} M^0 = N$ and $\operatorname{Tr} M^1 = 0$, so we may assume that $k \geqslant 2$. By definition, $\operatorname{Tr} M^k = \sum_{i_1,\ldots,i_k=1}^{N} M_{i_1 i_2} M_{i_2 i_3} \cdots M_{i_k i_1}$. Because the diagonal matrix elements of M vanish, we can restrict the above sum and consider only k-plet $(i_1, i_2, \ldots, i_k)$ such that $i_1 \neq i_2$, $i_2 \neq i_3, \ldots, i_{k-1} \neq i_k$, $i_k \neq i_1$. We call such k-plets *admissible*.

So, start with an admissible k-plet $I = (i_1, i_2, \ldots, i_k)$. To compute $\overline{M_{i_1 i_2} M_{i_2 i_3} \cdots M_{i_k i_1}}$ we argue as follows: the product $M_{i_1 i_2} M_{i_2 i_3} \cdots M_{i_k i_1}$ can take only two values, 0 or 1. It has value 1 if and only if each factor has value 1, that is if and only if $\{i_1, i_2\}, \{i_2, i_3\}, \ldots, \{i_k, i_1\}$ are edges of the graph with incidence matrix M. From our definition of probabilities on the space of incidence matrices, this happens with probability p^l, where l is the number of distinct pairs among $\{i_1, i_2\}, \{i_2, i_3\}, \ldots, \{i_k, i_1\}$. Obviously, l depends on I.

If we could find an efficient way to count the number of admissible k-plets I with given l, the problem would be solved. We have not been able to do so. However, there is a simple way, which we now expose, to group together families of admissible k-plets that are guaranteed to have the same l.

Fix a k-plet $W = (v_1, \ldots, v_k)$ of elements of $\{1, \ldots, k\}$ with the following properties: (i) $v_1 \neq v_2$, $v_2 \neq v_3, \ldots, v_{k-1} \neq v_k$, $v_k \neq v_1$, and (ii) if $v_\beta > 1$, there is a $\beta' < \beta$ such that $v_{\beta'} = v_\beta - 1$. Such a k-plet will be called a *normalized* k-plet in the sequel.

The first condition is almost the definition of an admissible k-plet, the only difference being that the members are in $\{1, \ldots, k\}$, not $\{1, \ldots, N\}$. The

second condition says that the order of appearance of elements of $\{1,..., k\}$ in the sequence $(v_1,..., v_k)$ is the natural order. Because of these two conditions, $v_1 = 1$ and $v_2 = 2$ in any normalized k-plet, but v_3 could be 1 or 3.

By condition (ii), the integers appearing in W build a set of the form $V = \{1,..., n\}$ for a certain $n \leqslant k$. Let E be the set whose elements are the (distinct) pairs among $\{v_1, v_2\}$, $\{v_2, v_3\}$,..., $\{v_k, v_1\}$.

Now choose an injective map σ from V to $\{1,..., N\}$. The number of such maps is $N^{\underline{n}} \equiv N(N-1)\cdots(N-n+1)$. Then the sequence $I = (\sigma(v_1),..., \sigma(v_k))$ is an admissible k-plet by the injectivity of σ and property (i) of the sequence W. Moreover, for the same reasons, the number l of distinct pairs among the k pairs $\{\sigma(v_1), \sigma(v_2)\}$, $\{\sigma(v_2), \sigma(v_3)\}$,..., $\{\sigma(v_k), \sigma(v_1)\}$ is exactly $|E|$, the number of elements of E. This number depends on W, but not on σ.

More precisely, there is a one to one correspondence between admissible k-plets I and pairs (W, σ). Note that the source of σ depends on W.

The bijection involves a simple but useful general algorithm, which we call the "label and substitute algorithm." We use it several times in the sequel. If $(O, O', O'',...)$ is any finite or infinite list of items (some items can be repeated), one can label the items in order of first appearance. This means that the first item receives label 1, then the next item different from the first one receives label 2 and so on. This gives a one to one map, the "labeling map." After that, by replacing each item in the list by its label, one obtains a sequence of integers, the "substitution sequence." It has the property that the first term is 1 and that if integer $i+1 \geqslant 2$ appears as a term, then integer i has appeared before. Note that this new sequence is invariant if we apply the "label and substitute algorithm" to it. Note also that the knowledge of the "labeling map" and the "substitution sequence" allows to reconstruct the original sequence. Let us give an example. The list (eat, work, eat, sleep, work, eat, work, sleep) leads to the "labeling map" (eat $\to$ 1, work $\to$ 2, sleep $\to$ 3), and to the "substitution sequence" (1, 2, 1, 3, 2, 1, 2, 3).

If $i = (i_1, i_2,..., i_k)$ is an admissible k-plet, we apply the "label and substitute algorithm." Then σ is the inverse of the "labeling map" of I and W is the "substitution sequence" of I. The properties of I making it an admissible k-plet and the definition of the "label and substitute algorithm" ensure that W is a normalized k-plet.

Written symbolically, this means that

$$\sum_I = \sum_W \sum_\sigma$$

where the sum over I is over admissible k-plets, the sum over W is over normalized k-plets, and the sum over σ is over injective maps as described

above. Inserting $\overline{M_{i_1 i_2} M_{i_2 i_3} \cdots M_{i_k i_1}}$ on both sides of this identity, we obtain our first important formula:

$$\overline{\mathrm{Tr}\, M^k} = \sum_W N^{\underline{|V|}} p^{|E|} \tag{2}$$

where V and E are functions of W as defined above. In this formula, the N and p-dependence are completely explicit. For finite k and large N this is clearly useful because the W's are defined independently of N and p.

It is not difficult in principle to enumerate normalized k-plets in standard lexicographic order, hence in particular in order of increasing $|V|$, and then compute for each normalized k-plet the value of $|E|$.

We know that

$$\overline{\mathrm{Tr}\, M^0} = N$$

and

$$\overline{\mathrm{Tr}\, M^1} = 0.$$

For $k = 2$, the only sequence is $(1, 2)$, so

$$\overline{\mathrm{Tr}\, M^2} = pN^{\underline{2}}.$$

For $k = 3$, the only sequence is $(1, 2, 3)$, so

$$\overline{\mathrm{Tr}\, M^3} = p^3 N^{\underline{3}}.$$

For $k = 4$, the sequences are $(1, 2, 1, 2)$, $(1, 2, 1, 3)$, $(1, 2, 3, 2)$ and $(1, 2, 3, 4)$, so

$$\overline{\mathrm{Tr}\, M^4} = p^4 N^{\underline{4}} + 2p^2 N^{\underline{3}} + pN^{\underline{2}}.$$

For $k = 5$, the sequences are $(1, 2, 1, 2, 3)$, $(1, 2, 1, 3, 2)$, $(1, 2, 1, 3, 4)$, $(1, 2, 3, 1, 2)$, $(1, 2, 3, 1, 3)$, $(1, 2, 3, 1, 4)$, $(1, 2, 3, 2, 3)$, $(1, 2, 3, 2, 4)$, $(1, 2, 3, 4, 2)$, $(1, 2, 3, 4, 3)$, and $(1, 2, 3, 4, 5)$, so

$$\overline{\mathrm{Tr}\, M^5} = p^5 N^{\underline{5}} + 5p^4 N^{\underline{4}} + 5p^3 N^{\underline{3}}.$$

For $k = 6$, one finds 41 sequences leading to

$$\begin{aligned}\overline{\mathrm{Tr}\, M^6} = {} & p^6 N^{\underline{6}} + (3p^6 + 6p^5)\, N^{\underline{5}} \\ & + (9p^5 + 6p^4 + 5p^3)\, N^{\underline{4}} + (4p^3 + 6p^2)\, N^{\underline{3}} + pN^{\underline{2}}.\end{aligned}$$

For $k \geqslant 7$, by counting the sequences with $|V| \geqslant k-2$,

$$\begin{aligned}
\overline{\mathrm{Tr}\, M^k} &= p^k N^{\underline{k}} + \left\{ \left(\frac{k^{\underline{2}}}{2} - 2k \right) p^k + k p^{k-1} \right\} N^{\underline{k-1}} \\
&\quad + \left\{ \left(\frac{k^{\underline{4}}}{8} - \frac{5k^{\underline{3}}}{6} + k^{\underline{2}} + 5k \right) p^k + \left(\frac{k^{\underline{3}}}{2} - k^{\underline{2}} - 6k \right) p^{k-1} \right. \\
&\quad \left. + \left(\frac{k^{\underline{2}}}{2} + k \right) p^{k-2} \right\} N^{\underline{k-2}} + O(N^{k-3}) \\
&= p^k N^k + (-2kp^k + kp^{k-1})\, N^{k-1} \\
&\quad + \left\{ (k^2 + 4k)\, p^k - (k^2 + 5k)\, p^{k-1} + \frac{1}{2} (k^2 + k)\, p^{k-2} \right\} N^{k-2} \\
&\quad + O(N^{k-3}).
\end{aligned}$$

With ten days of computation on a workstation, we have obtained the number of normalized k-plets with given $|V|$ and $|E|$ up to $k = 18$. The results are available upon request to the authors.

3.2. Sum Rule for $p = 1$

We have checked our enumeration against a simple sum rule. We put $p = 1$ and sum over $|E|$. In this case, with probability 1, the random graph becomes complete and the matrix M is equal to $J - Id$, where Id is the $N \times N$ identity matrix and J is the $N \times N$ matrix with all entries equal to 1. But $J - Id$ has only two eigenvalues, $N-1$ with multiplicity 1 and -1 with multiplicity $N-1$. So, for $p = 1$,

$$\overline{\mathrm{Tr}\, M^k} = \mathrm{Tr}(J - Id)^k = (N-1)^k + (N-1)(-1)^k.$$

The general formula reduces to

$$(N-1)^k + (N-1)(-1)^k = \sum_{W} N^{\underline{|V|}} = \sum_{n} N^{\underline{n}} \mathscr{D}_{k,n}$$

where $\mathscr{D}_{k,n}$ is the number of normalized k-plets W with $|V| = n$. Going to generating functions, the left-hand side gives

$$\sum_{k,N} ((N-1)^k + (N-1)(-1)^k) \frac{x^N}{N!} \frac{t^k}{k!} = e^{-t}(e^{xe^t} + e^x(x-1))$$

while the right-hand side gives

$$\sum_{k,N} \frac{x^N}{N!} \frac{t^k}{k!} \sum_n N^{\underline{n}} \mathscr{D}_{k,n} = e^x \sum_{k,n} \mathscr{D}_{k,n} x^n \frac{t^k}{k!}.$$

Hence

$$\sum_{k,n} \mathscr{D}_{k,n} x^n \frac{t^k}{k!} = e^{-t}(e^{x(e^t-1)} + x - 1).$$

By using any symbolic computation software, the computation of all the $\mathscr{D}_{k,n}$'s up to, say, $k = 50$ takes only a few seconds. We can express $\mathscr{D}_{k,n}$ in terms of standard Stirling numbers of the second kind, $\mathscr{S}_{k,n}$. We shall meet them again in Section 5.5. They can be characterized by the relation $x^k = \sum_n \mathscr{S}_{k,n} x^{\underline{n}}$. Using the trick $\overline{\mathrm{Tr}\, M^{k+1}} + \overline{\mathrm{Tr}\, M^k} = N(N-1)^k = \sum_n \mathscr{S}_{k,n} N^{\underline{n+1}}$, one finds $\mathscr{D}_{k+1,n} + \mathscr{D}_{k,n} = \mathscr{S}_{k,n-1}$, which gives for $k \geqslant 1$

$$\mathscr{D}_{k,n} = \sum_{r=1}^{k-1} (-1)^{k-1-r} \mathscr{S}_{r,n-1}.$$

For $x = 1$, we get the generating function of $\mathscr{D}_k = \sum_n \mathscr{D}_{k,n}$, the total number of normalized k-plets. That is $e^{(e^t-1-t)}$ and we recognize that the $\mathscr{D}_k$ are so-called generalized Bell numbers.[23] To give an idea of the size of the computations of the first moments, the values of $\mathscr{D}_k$ for $k = 0,..., 18$ are 1, 0, 1, 1, 4, 11, 41, 162, 715, 3425, 17722, 98253, 580317, 3633280, 24011157, 166888165, 1216070380, 9264071767 and 73600798037. With our combinatorial interpretation of $\mathscr{D}_k$, it is clear that $\mathscr{D}_k \leqslant k!$. On the other hand, the saddle point evaluation shows that $\log \mathscr{D}_k \propto k(\log k + o(\log k))$, confirming that the growth of the computation is extremely rapid.

3.3. Model with Random Signs

The above calculations could be adapted to the variant of the model (see Section 2) defined by a parameter $a \in [0, 1]$ where the non-zero elements of the random matrix are $+1$ with probability a and -1 with probability $1-a$. Equivalently the edges of the random graph are dressed with a random sign ± 1.

For an admissible k-plet $I = (i_1, i_2,..., i_k)$, we find $\overline{M_{i_1 i_2} M_{i_2 i_3} \cdots M_{i_k i_1}} = p^{l_e}(pb)^{l_o} = p^l b^{l_o}$ where $b = 2a - 1$ is the sign asymmetry, $l = l_e + l_o$ is the number of distinct pairs among $\{i_1, i_2\}, \{i_2, i_3\},..., \{i_k, i_1\}$ and l_e (resp. l_o)

the number of those which are repeated an even (resp. odd) number of times. So, Eq. (2) becomes

$$\overline{\mathrm{Tr}\, M^k} = \sum_W N^{\underline{|V|}} p^l b^{l_o} \tag{3}$$

and the first moments can be exactly computed by enumerating all the normalized k-plets:

$$\overline{\mathrm{Tr}\, M^2} = pN^{\underline{2}},$$

$$\overline{\mathrm{Tr}\, M^3} = p^3 b^3 N^{\underline{3}},$$

$$\overline{\mathrm{Tr}\, M^4} = p^4 b^4 N^{\underline{4}} + 2p^2 N^{\underline{3}} + pN^{\underline{2}},$$

$$\overline{\mathrm{Tr}\, M^5} = p^5 b^5 N^{\underline{5}} + 5p^4 b^3 N^{\underline{4}} + 5p^3 b^3 N^{\underline{3}},$$

$$\overline{\mathrm{Tr}\, M^6} = p^6 b^6 N^{\underline{6}} + (3p^6 b^6 + 6p^5 b^4)\, N^{\underline{5}} + (9p^5 b^4 + 6p^4 b^4 + 5p^3)\, N^{\underline{4}} + (4p^3 + 6p^2)\, N^{\underline{3}} + pN^{\underline{2}}.$$

Of course, the case $a = 1$ gives previous results. If we concentrate on the even model with $a = 1/2$ (for which $b = 0$), we see that the summation over random signs keeps only the walks for which all the edges are visited an *even* number of times. Consequently for all the odd moments, $\overline{\mathrm{Tr}\, M^{2k+1}} = 0$ and the density of states is a symmetric distribution.

3.4. Generating Function

While very convenient for explicit enumeration, the formula in Eq. (2) is not always convenient for theoretical arguments. So we reformulate it.

Starting from a normalized k-plet W, we have defined two sets V and E. Recall that V is of the form $\{1,..., n\}$ for some n and that E is made of pairs of distinct elements of V. These are exactly the data for a labeled graph with vertex set V and edge set E. In this framework, W can be interpreted as a closed walk on the graph visiting all edges (this implies in particular that the graph is connected), the order of first visit to a vertex respecting the natural order: W starts at vertex 1, and its first visit to vertex 2 occurs before its first visit to vertex 3 and so on. Note that in this formulation, many labeled graphs do not appear (for example, the labeled graphs for which vertices 1 and 2 are not linked by an edge). To resume, only some labeled graphs and some walks visiting all edges appear. On the other hand, take an unlabeled connected graph G isomorphic to the one defined by the normalized sequence W. Clearly, it is possible to label it in

such a way that W describes a closed walk on G visiting all edges. This labeling can be achieved exactly in $|\mathrm{Aut}(G)|$ (the order of the automorphism group of G, see Section 2) ways. Indeed, two distinct labelings have to describe a non-trivial automorphism of G because W determines completely a labeled graph isomorphic to G. Written symbolically, this means that

$$\sum_W = \sum_G \frac{1}{|\mathrm{Aut}(G)|} \sum_{W(G)}$$

where on the left-hand side the summation is over normalized k-plets whereas on the right-hand side the summation on G is over isomorphism classes of connected graphs and the summation on $W(G)$ is over walks on G of length k visiting all edges of G. So, the fundamental identity can be rewritten as

$$\overline{\mathrm{Tr}\, M^k} = \sum_G \frac{1}{|\mathrm{Aut}(G)|} N^{\underline{|V(G)|}} p^{|E(G)|} W_k(G) \tag{4}$$

where the sum over G is over isomorphism classes of connected graphs, $\mathrm{Aut}(G)$ is the automorphism group of G, $V(G)$ and $E(G)$ are respectively the vertex set and the edge set of G and $W_k(G)$ is the number of closed walks of k steps on G visiting all edges of G. Note that graphs with $|V(G)| > N$ or $|E(G)| > k$ do not contribute. Although Eq. (4) was established for $k \geqslant 2$, it is valid for $k \geqslant 0$. For $k = 1$, as a walk with only one step cannot be closed, $W_1(G) = 0$ and $\overline{\mathrm{Tr}\, M} = 0$. For $k = 0$, as a walk with zero steps is closed and visits one vertex and zero edges, $W_0(G) = 0$ for every graph except for $\bullet$, the graph with one vertex. For this graph, $W_0(\bullet) = 1$, leading to $\overline{\mathrm{Tr}\, M^0} = N$. By convention, the empty graph is not counted as connected, so it does not appear in Eq. (4).

The above formula can be used to build a generating function by summing over k. On the left-hand side $\sum_{k \geqslant 0} \lambda^k \overline{\mathrm{Tr}\, M^k} = \overline{\mathrm{Tr}(1 - \lambda M)^{-1}}$ is a rational function of λ because it is the average of a finite number (namely $2^{N(N-1)/2}$) of rational functions. To deal with the right-hand side, define $W_G(\lambda) = \sum_k \lambda^k W_k(G)$. This counts closed walks on G of arbitrary length visiting all edges of G. Were it not for the last constraint, life would be easy because the generating function for the closed walks on G of arbitrary length is simply $\mathrm{Tr}(1 - \lambda G)^{-1}$ (in this formula and in later algebraic expressions involving graphs and traces, we use a convenient abuse of notation: G denotes at the same time the graph and the incidence matrix obtained by labeling it, any choice of labeling leads to the same traces). To

suppress walks that do not visit all edges, we can use inclusion-exclusion to obtain:

$$W_G(\lambda) = \mathrm{Tr}\,\frac{1}{1-\lambda G} - \sum_{G^{(1)}} \mathrm{Tr}\,\frac{1}{1-\lambda G^{(1)}} \cdots (-)^l \sum_{G^{(l)}} \mathrm{Tr}\,\frac{1}{1-\lambda G^{(l)}} \cdots \tag{5}$$

where $\sum_{G^{(l)}}$ is the sum over all subgraphs of G obtained by deleting l edges, so this formula ends at $l = |E(G)|$. This expresses $W_G(\lambda)$ as a finite sum of rational functions, so $W_G(\lambda)$ is a rational function. To summarize, we have proved an identity between rational functions (so that it is possible to assign values to λ):

$$\overline{\mathrm{Tr}\,\frac{1}{1-\lambda M}} = \sum_G \frac{1}{|\mathrm{Aut}(G)|} N^{\underline{|V(G)|}} p^{|E(G)|} W_G(\lambda). \tag{6}$$

4. FIXED EDGE PROBABILITY *P*

In the large N limit with p fixed, Füredi and Komlós[13] (following work by Wigner,[26] Arnold[1] and Juhász[17]) have given a detailed description of the spectrum: it consists $N-1$ "small" eigenvalues which after a rescaling by a factor $\sqrt{p(1-p)\,N}$ build up a semicircle distribution of radius 2 and one large eigenvalue (the Perron eigenvalue) whose distribution is Gaussian with average $pN + (1-2p)$ and finite variance $2p(1-p)$. The appearance of this isolated eigenvalue is due to the non-vanishing average of the matrix elements.

We have used this theorem to check our numerical simulations and our algorithm for computing the moments. We have also computed numerically level spacings. They obey the Gaussian Orthogonal Ensemble statistics with good accuracy.

4.1. Monte-Carlo Simulations

Monte-Carlo simulations consist in generating a lot of incidence matrices of random graphs, computing and studying their spectra.

A matrix is obtained with the following procedure. It is a symmetric matrix M of size $N \times N$. Its diagonal elements are set to 0. Its non-diagonal elements $M_{ij} = M_{ji}$ (with $i \neq j$) represent the edges of the graph. They are randomly and independently chosen: their values are 1 with probability p and 0 with probability $1-p$, where p is a fixed parameter.

As the matrix is symmetric, it is diagonalizable and all its eigenvalues are real numbers. They are computed using the appropriate routine in Nag library. This procedure is repeated with several random matrices. As we are

interested by the asymptotic behavior when N is large, we compare different sizes of matrix: $N = 1000$, 2000 and 4000, with 40, 20 and 10 matrices respectively. So, for each case, the statistical study is done over 40000 random eigenvalues.

When the edge probability p is fixed, in the spectrum of a given matrix, we must make the distinction between its largest eigenvalue and the $N-1$ others. Indeed, all the elements are non-negative ($M_{ij} \geqslant 0$). Moreover, when N is not too small, the random graph is connected and the matrix is irreducible. Then, the Perron–Frobenius theorem[14] assures that the eigenvalue with the largest modulus (the Perron eigenvalue) is non-degenerate, positive, and that the elements of the corresponding eigenvector are all positive.

Numerical observations show with great accuracy that the average of this Perron eigenvalue is pN, plus a finite correction which depends on p. Furthermore its variance (i.e., the square of the width of its distribution) is also finite et depends on p. Indeed, for large N, the Perron eigenvector is equal to $(1 \cdots 1)^T$, plus small fluctuations, and the eigenvalue is about pN.

As the rest of the spectrum has a large N behavior which is different we eliminate, in the rest of this section, the largest eigenvalue of each matrix. On Fig. 1, histograms of eigenvalues for several values of p are shown. To allow comparison, the eigenvalues (x-axis) have been divided by $\sqrt{p(1-p)\,N}$, and the y-coordinates have been scaled in order to normalize the area of each histogram. We see clearly that the asymptotic shape of

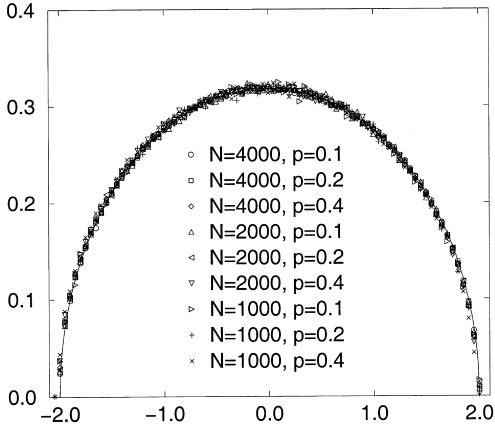

Fig. 1. Histograms of spectra of random graph incidence matrices, for several sizes N of matrices and several edge parameters p. The x-axis is rescaled by $\sqrt{p(1-p)\,N}$ and the area of each histogram is normalized. All the curves coincide with the semicircle of radius 2, drawn with a solid curve.

the rescaled distribution is a semicircle of radius 2. In particular, only the variance (or the width) of the distribution depends on N and p, but the shape remains the same. So already for $N \simeq 1000$ the agreement with the large N estimates in the Füredi–Komlós theorem is very good.

The nearest neighbor spacing distribution is commonly studied to observe fluctuations in random spectra. If $(\lambda_1, \lambda_2, ..., \lambda_N)$ are the eigenvalues in ascending order of a given random incidence matrix, we define the normalized spacings as

$$s_i = \frac{1}{4}\sqrt{\frac{N}{p(1-p)}}\,(\lambda_{i+1} - \lambda_i)$$

in order to have $\langle s \rangle = 1$, by omitting the last spacing s_{N-1} which involves the Perron eigenvalue. On Fig. 2, histograms of normalized spacings are shown. We see clearly that they have the same shape. This indicates that the large N behavior is independent of p.

To allow the comparison with the spacing distribution of the Gaussian Orthogonal Ensemble (GOE) of random matrices, we use the "Wigner surmise"

$$q_0(s) = \frac{\pi}{2}\, s \exp\left(-\frac{\pi}{4}s^2\right).$$

It is considered to be an excellent approximation of the GOE distribution.[19] However, a direct comparison between the Monte-Carlo histograms

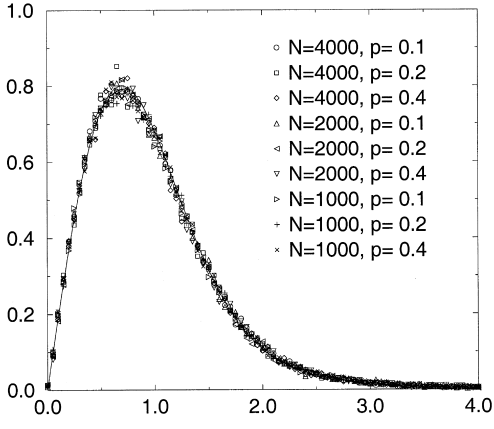

Fig. 2. Histograms of normalized nearest neighbor spacings in spectra of random graph incidence matrices, for several sizes N of matrices and several edge parameter p. All the curves coincide with the GOE spacing distribution distorted by the semicircle law, drawn with a solid curve.

and $q_0(s)$ would give bad results. Indeed, $q_0(s)$ describes the distribution of the *locally* normalized spacings $N\rho(\lambda_i)(\lambda_{i+1}-\lambda_i)$, where $\rho(\lambda)$ is the eigenvalue probability distribution. So the probability distribution of $s_i = N(\lambda_{i+1}-\lambda_i)$ is

$$q(s) = \int d\lambda\, \rho(\lambda)^2\, q_0(\rho(\lambda)\, s).$$

By taking the semicircle distribution $\rho(\lambda) = 4/\pi\sqrt{1-4\lambda^2}$ with a diameter 1 in order to have $\langle s\rangle = 1$,

$$\begin{aligned} q(s) &= \frac{6}{\pi}\, s \exp\left(\frac{-4}{\pi}s^2\right) F\left(\frac{1}{2}, 3, \frac{4}{\pi}s^2\right) \\ &= \frac{12}{\pi}\, s \exp\left(\frac{-4}{\pi}s^2\right) \sum_{k=0}^{\infty} \frac{\Gamma(k+1/2)}{\Gamma(1/2)} \frac{1}{k!(k+2)!} \left(\frac{4}{\pi}s^2\right)^k \end{aligned}$$

where $F(1/2, 3, z)$ is a generalized hypergeometric function. On Fig. 2, we see that the Monte-Carlo simulations coincide with the function $q(s)$, giving good evidence that the incidence matrices of random graphs are, as expected, in the universality class of GOE.

4.2. Perturbative Expansion for the Perron Eigenvalue

We retrieve quickly the main features of the distribution of the Perron (i.e., largest) eigenvalue via a perturbative expansion. By definition, $\bar{M} = p(J - Id)$ where Id is the $N \times N$ identity matrix and J is $N \times N$ matrix with all entries equal to 1, that is, N times the projector on $|\Omega\rangle = (1/\sqrt{N})(1 \cdots 1)^T$. So $|\Omega\rangle$ is the Perron eigenvector of $\bar{M}$ with eigenvalue $p(N-1)$. We define $D = M - \bar{M}$. Simple manipulations[25] show that an eigenvalue λ of M whose eigenspace is one-dimensional and not orthogonal[3] to $|\Omega\rangle$ satisfies

$$1 = pN \langle \Omega | \frac{1}{\lambda + p - D} |\Omega\rangle.$$

The perturbative expansion in D gives for the Perron eigenvalue

$$\lambda = p(N-1) + \langle \Omega | D |\Omega\rangle + \frac{1}{pN}(\langle \Omega | D^2 |\Omega\rangle - \langle \Omega | D |\Omega\rangle^2) + \cdots.$$

[3] This explain why perturbation theory in D singles out a particular eigenvalue, the Perron eigenvalue, the only eigenvalue not orthogonal to $|\Omega\rangle$ to order 0 in D.

Explicit computation yields

$$\overline{\langle \Omega |\, D\, | \Omega \rangle} = 0$$

$$\overline{\langle \Omega |\, D^2\, | \Omega \rangle} - \overline{\langle \Omega |\, D\, | \Omega \rangle^2} = p(1-p)(N-1)(N-2)/N.$$

Hence

$$\overline{\lambda^k} = p^k N^k \left\{ 1 + \frac{k}{pN}(1-2p) + O\left(\frac{1}{N^2}\right) \right\}.$$

This is enough to show that the distribution of the Perron eigenvalue has average $\bar{\lambda} = pN + 1 - 2p + O(1/N)$ and *finite* width $\overline{\lambda^2} - \bar{\lambda}^2 = O(1)$. Let us call $\overline{\mathrm{tr}'\, M^k}$ the kth moment of the distribution of *other* eigenvalues,

$$\overline{\mathrm{tr}'\, M^k} \equiv \frac{1}{N-1}\left(\overline{\mathrm{Tr}\, M^k} - \overline{\lambda^k}\right).$$

Comparison of our formulae for $\overline{\mathrm{Tr}\, M^k}$ and $\overline{\lambda^k}$ leads to

$$\overline{\mathrm{tr}'\, M} = -p + 2p/N + O(1/N^2),$$

$$\overline{\mathrm{tr}'\, M^2} = p(1-p)\, N + p(3p-2) + O(1/N),$$

$$\overline{\mathrm{tr}'\, M^3} = -3p^2(1-p)\, N + O(1),$$

$$\overline{\mathrm{tr}'\, M^4} = 2[\, p(1-p)\,]^2\, N^2 + O(N),$$

$$\overline{\mathrm{tr}'\, M^k} = O(N^{k-3}) \qquad \text{for} \quad k \geqslant 5.$$

This is of course consistent with ref. 13. Note however that for the Laplacian matrix (see Appendix A) the hypotheses of the theorem are not fulfilled because the diagonal elements are correlated to the rest of the matrix and have a variance of order N. And indeed, the Laplacian has an entirely different spectral distribution.

4.3. Comparison with the Model with Random Signs

The random sign model defined in Section 2 (where $M_{ij} = +1$ with probability ap, -1 with probability $(1-a)\, p$ and 0 with probability $1-p$) is also covered by the results in ref. 13. If $a \neq 1/2$, the "small" eigenvalues build a semicircle of radius 2 after rescaling by $\sqrt{p(1-b^2p)\, N}$ (where, as before, $b = 2a - 1$), and the large eigenvalue has a Gaussian distribution with average $(N-1)\, bp + (1-b^2p)/b$ and variance $2p(1-b^2p)$.

As another check of our formulae, we give a short proof of the semicircle distribution for the symmetric ($a = 1/2$) random sign model, where $M_{ij} = \pm 1$ with probability $p/2$ and 0 with probability $1 - p$. To compute $\overline{\mathrm{Tr}\, M^k}$ from Eq. (3), we must keep in the summation only the W's for which the edges are repeated an even number of times. Then for the odd moments, $\overline{\mathrm{Tr}\, M^{2k+1}} = 0$. For even moments, $\overline{\mathrm{Tr}\, M^{2k}}$, the maximal number of distinct edges in W is k. Moreover in the large N limit with p fixed, we retain the W's with the maximal number of vertices. This maximum is $k + 1$ and is obtained with W's associated to rooted planar trees with k edges, as explained later in Section 5.5. Then, for $a = 1/2$,

$$\frac{1}{N}\overline{\mathrm{Tr}\, M^{2k}} = C_k p^k N^k + O(N^{k-1})$$

where C_k are Catalan numbers, $C_k = (2k)!/[k!\,(k+1)!]$. As the Catalan numbers are the moments of the semicircle law of radius 2, this shows that the density of states is the semicircle law of radius $2\sqrt{pN}$.

5. FIXED AVERAGE CONNECTIVITY α

In this section, we present results for the large N limit with $\alpha = pN$ fixed. After displaying numerical observations, we prove that the density of states has an infinity of delta peaks for any α. Then we explain how to compute the $2k$th moment of the density of states which is a polynomial in α of degree k. Finally, we give bounds for the coefficients of these polynomials and use the bounds to show that the spectrum is determined by the moments.

5.1. Monte-Carlo Simulations and Observations

In Section 2, we have defined a variant of the model where the nonzero elements of the random matrix have a random sign: $+1$ with probability a and -1 with probability $1 - a$. We will prove in Section 5.3 that in the large N limit with α fixed, the moments of the density of states are independent of a. Then it is expected that the density of states evaluated by our Monte-Carlo simulations are very similar for those equivalently obtained[11, 12] for the even model with $a = 1/2$. However as we will clarify some points, we have repeated these simulations.

Monte-Carlo simulations have been done with the same procedure as in Section 4.1. We have seen that, for a fixed edge probability p, the shape of the distribution of eigenvalues is a semicircle. But when p becomes of

order $1/N$, the distribution gets strongly distorted. To compare different sizes N, let us fix $\alpha = pN$. It is the average number of 1's in a given row (or column) of the random matrix. For the graph, it is the averaged connectivity (i.e., the average number of neighbors of a given vertex).

On Fig. 3, histograms of eigenvalues for several values of α are shown. For each value of α, three sizes of matrices, $N = 1000$, 2000 and 4000, have been simulated with 40, 20 and 10 matrices respectively. So, for each case, the statistical study is done over 40000 random eigenvalues. For fixed α, the three curves are superposed, in the limits of Monte-Carlo fluctuations. So we can consider that we observe the asymptotic distribution $d\rho_\alpha$ for large N, which depends only on α. To allow comparison between different α's, the eigenvalues (x-axis) have been divided by $\sqrt{\alpha}$, and the y-coordinates have been scaled in order to normalize the area of each histogram.

When α is small, we see a forest of delta peaks. This has been previously observed in other models of sparse random matrices.[10, 18] Their heights are not representative with this kind of histogram because they

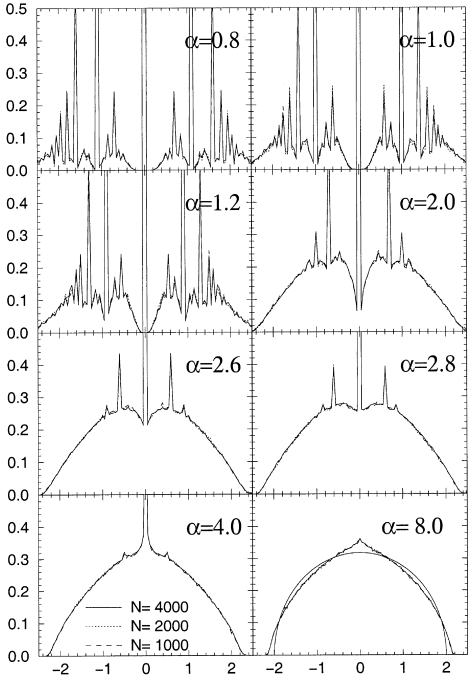

Fig. 3. Histograms of spectra of random graph incidence matrices, for different sizes N of matrices and probability parameter $p = \alpha/N$, with α fixed. The x-axis is rescaled by $\sqrt{\alpha}$ and the area of each histogram is normalized. For comparison, the semicircle is drawn for $\alpha = 8$.

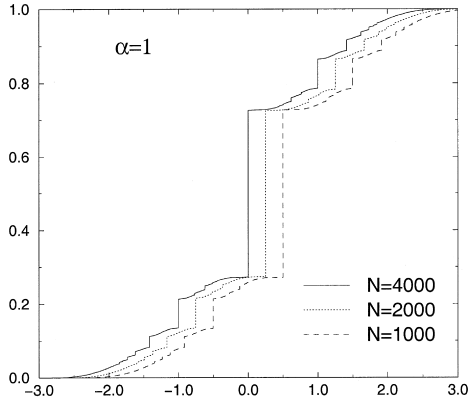

Fig. 4. Cumulative histograms of random graph incidence matrices, for different sizes N of matrices and probability parameter $p = \alpha/N$, with $\alpha = 1$. Each vertical step represents a delta peak. As the three curves are quite similar, they are shifted along the x-axis to be more visible. The largest step is at $x = 0$.

depend on the width of the bin, chosen arbitrarily. To give a correct representation of the importance of delta peaks, the cumulative distribution function (i.e., the integral of the distribution from $-\infty$) is better. We plot it for $\alpha = 1$ on Fig. 4: each vertical step corresponds to a delta peak, with a weight equal to the height of the step. As the heights are comparable between the different sizes, the delta peaks survive in the limit $N \to \infty$.

We observe that for any α the bigger delta peaks are, in order of importance, at $x = 0$, ± 1, $\pm\sqrt{2}$, $\pm(\sqrt{5} \pm 1)/2$ (golden mean), $\pm\sqrt{3}$, $\pm\sqrt{2 \pm \sqrt{2}}$, etc. These values can be recognized as eigenvalues of small trees.

When α increases, the heights of delta peaks decreases, but their positions along the x-axis do not move (before the $\sqrt{\alpha}$ rescaling). With this kind of histogram, when a delta peak becomes too small, it seems to disappear because it is drowned in the rest of the distribution. But we will show that the spectrum has an infinity of delta peaks for any α. For large values of α, the shape is close to the semicircle, obtained when p is finite (which corresponds to the limit $\alpha = \infty$).

In Section 2, we have seen that the topology of random graphs has a percolation transition at $\alpha = 1$. But, this transition seems without effects for the distribution of eigenvalues: on Fig. 3, distribution for $\alpha = 0.8$, 1 and 1.2 are qualitatively similar. In particular, we have checked that the height of the delta peak at $x = 0$ is regular around $\alpha = 1$. The same remarks apply to the localization transition, conjectured in refs. 11 and 12 to be for $\alpha \approx 1.4$.

In contrast, we observe a change of behavior between $\alpha = 2.6$ and 2.8, for the distribution $d\rho_\alpha$ in the vicinity of $x = 0$. For $\alpha \leqslant 2.6$ (resp. $\geqslant 2.8$),

$d\rho_\alpha(\lambda)$ decreases (resp. increases) when λ goes to 0^+. It is difficult to say if the limit $d\rho_\alpha(\lambda)$ when λ goes to 0^+ is 0 or not in the small α phase, and ∞ or not in the large α phase. This transition seems to be related to a transition at $\alpha = e$, where the second derivative of the height of delta peak at $x = 0$ as function of α is discontinuous.[3] Unfortunately, this discontinuity is too small to be seen in our Monte-Carlo simulations.

We have also studied the distribution of nearest neighbors spacings. Our conclusions are similar to those of Evangelou and Economou:[11, 12] for small α, the distribution numerically coincides with an exponential, and for large α with the GOE law. In the vicinity of the localization transition, $\alpha \approx 1.4$, the distribution interpolates between these two forms. On the other hand, we have not been able to reproduce results[11, 12] concerning the singularity of the spectral distribution as λ goes to 0.

5.2. Existence of Delta Peaks

In this section, we explain that, for any value of α, the distribution has an infinite number of delta peaks. More precisely, we show that they are delta peaks at all eigenvalues of finite trees. However their heights are exponentially decreasing functions of α; most of them are hidden in simulations by the statistical noise and the distribution seems to be quite smooth for large α.

The delta peaks at tree eigenvalues have, at least, two origins:[18] the small connected components, which are trees, and small trees grafted on the giant component (percolation cluster).

The random incidence matrix is block-diagonal, with one block per each connected component. As shown by Eq. (1), the average number of times a given tree T appears as a connected component of the random graph is proportional to N. So, for any eigenvalue of the incidence matrix of any tree with n vertices, a contribution of height $\alpha^{n-1}e^{-n\alpha}/|\mathrm{Aut}(T)|$ to the corresponding delta peak appears. These eigenvalues are algebric numbers, i.e., solutions of a polynomial equation with integer coefficients of degree at most n. The height (but not the position) of the peak depends on α. The height decreases exponentially with n, so only eigenvalues of small trees appear with repetitions in Monte-Carlo simulations.

Furthermore, for $\alpha > 1$, even the giant component contributes to delta peaks. To explain why, we first define grafting. Let $G = (V, E)$ be a graph, $G' = (V', E')$ and $G'' = (V'', E'')$ be two subgraphs of G, and V_0 be a subset of V. We say that G is obtained by grafting G' on G'' along V_0 if $V_0 = V' \cap V''$ and E is the disjoint union of E' and E''.

If this is the case, suppose moreover that the incidence matrix of G' has an eigenvector ϕ' (with eigenvalue λ) whose components on V_0 are 0.

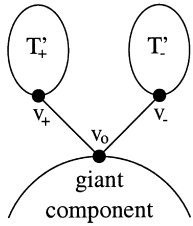

Fig. 5. A symmetric graft on the giant component gives delta peaks in the distribution of eigenvalues.

Then the vector ϕ obtained by extending ϕ' to V by 0 is an eigenvector of G with the same eigenvalue λ.

Now if G is the giant component of a random graph, if G' has a finite number of vertices and is connected, then G' is a *tree* with probability 1. Indeed, it is known[6] that finite connected subgraphs with loops are suppressed by powers of N^{-1}, even if they belong to the giant component. For a given tree T, the average number of times G can be obtained by grafting a subgraph G' isomorphic to T on a subgraph G'' along n points is of order $N\alpha^{|E(T)|}e^{(n-V(T))\alpha}/|\mathrm{Aut}(T)|$ for large α. So we have shown that if T has an eigenvector (with eigenvalue λ) vanishing on some vertices, λ appears as a delta peak in the spectrum of the giant component.

Now if λ is an eigenvalue of the incidence matrix of a tree T' with eigenvector ϕ', then λ also appears as an eigenvalue of some tree T whose associated eigenvector vanishes on some vertices of T: for instance, choose a vertex v on T', take two copies T'_+ and T'_- of T' and built a tree T by joining v_+ and v_- to a new vertex v_0. Then $\phi = (\phi', 0, -\phi')$ on $V(T) = V(T'_+) \cup \{v_0\} \cup V(T'_-)$ is an eigenvector of T with eigenvalue λ and this one can be grafted along v_0 (see Fig. 5). So we have shown that the giant component contributes to delta peaks in the spectrum at all eigenvalues of finite trees.

To summarize, at any eigenvalue of any finite tree, a delta peak appears in the distribution, with contributions both from small connected components and from the giant component (for $\alpha > 1$). We do not know if delta peaks appear at other positions, due to other mechanisms. On the other hand, we have not been able to prove that for $\alpha > 1$ the giant component gives a continuous part in the distribution of eigenvalues.

5.3. General Considerations

From an analytical viewpoint, we use Eq. (2) which we rewrite as

$$\overline{\mathrm{Tr}\, M^k} = \sum_W N^{\underline{|V|}} N^{-|E|} \alpha^{|E|}$$

and recall that on the right-hand side, E and V can be interpreted as vertices and edges of a connected graph. It is then an elementary topological fact that $|E| \geqslant |V| + 1$, with equality if and only if the graph is a tree. For fixed k, the possible graphs form a finite set, and there is no difficulty to take the large N limit:

$$\mu_k \equiv \lim_{N \to \infty} \frac{1}{N} \overline{\mathrm{Tr}\, M^k} = \sum_W \alpha^{|E|}$$

where the sum of over normalized k-plets W associated to trees. The existence of the $N \to \infty$ limit above is a strong indication of the existence of a limit eigenvalue probability density $d\rho_\alpha$, for which μ_k is just the usual kth moment.

In the same way, in the large N limit with fixed α, Eq. (6) becomes

$$\lim_{N \to \infty} \frac{1}{N} \overline{\mathrm{Tr} \frac{1}{1 - \lambda M}} = \sum_T \frac{1}{|\mathrm{Aut}(T)|} \alpha^{|E(T)|} W_T(\lambda) \tag{7}$$

where the sum runs over the trees T. For trees, we observe that the computation of $W_T(\lambda)$ using Eq. (5) can be simplified: the sum over all subgraphs of T can be reduced[2] to the sum over all subtrees obtained by deleting leaves of T.

The dominance of trees here and in Section 2 has a similar origin. Though the giant component contains of order N loops, only a finite number of them are finite: as already noticed before, a finite connected induced subgraph of a random graph at fixed α is a tree with probability 1.

On a tree, it takes an even number of steps to make a closed walk, and $\mu_k = 0$ for odd k. This elementary observation implies in fact that trees can be bicolored, and by standard argument, this shows that trees have a symmetric spectrum (see Section 5.6 for an application of this property). As random graphs look locally like trees, it is not too surprising that they also have a symmetric spectrum (this is of course true only for $N \to \infty$). Later we shall show that the moments μ_k determine the distribution. Then $\mu_k = 0$ for odd k implies that the distribution of eigenvalues is indeed symmetric.

For the even moments, μ_{2k} is a polynomial of α with degree k,

$$\mu_{2k} = \sum_l \mathscr{I}_{k,l} \alpha^l$$

where $\mathscr{I}_{k,l}$ is the number of normalized $2k$-plets W associated to trees with l edges.

If we consider the variant of the model with random signs (see Section 2), the same arguments apply. In particular, only W's associated to trees

contribute to the sums. But in this case, any edge is visited an even number of times and, following Eq. (3), μ_{2k} does not depend to a. Then, in the large N limit with α fixed, the parameter a is irrelevant.

5.4. Recursion Relation for $\mathscr{I}_{k,l}$

Consider a normalized $2k$-plet W associated to a tree T, with $k \geqslant 1$. Recall that W induces a walk covering T, hence a labeling of T (vertices are labeled in order of their appearance in the walk). Let r be the root of T, the vertex labeled 1, where the walk starts. There is always an edge between vertices labeled 1 and 2. If the edge $\{1, 2\}$ is cut, the tree breaks in two trees, T' whose root r' is vertex 2 and T'' whose root r'' is vertex 1 (note that $r'' = r$). The tree T can be seen as trees T' and T'' linked by the edge $\{r'', r'\}$, as shown in Fig. 6. Note that T' and T'' are arbitrary trees. They could for instance consist of a single vertex.

The walk W can be decomposed accordingly: the walker starts at vertex r'', walks along edge $\{r'', r'\}$, makes a closed walk on T' (made possibly of zero steps), walks along edge $\{r', r''\}$, makes a closed walk on T'' (made possibly of zero steps), and so on and so forth, and finally comes back to vertex r'', after covering T' and T''. Let $2k'$ (resp. $2k''$) be the number of steps made on T' (resp. T''). We can glue the small pieces of walks on T' (resp. T'') together into a single walk, covering T' (resp. T'') because W covers T. This walk is a sequence of vertices on T' (resp. T''), and the "label and substitute algorithm" applied to this sequence leads to a normalized $2k'$-plet W' (resp. to a normalized $2k''$-plet W'').

Now, the $2k$-plets W giving rise to the same pair (W', W'') are easily counted. They differ from each other by the organization of the steps on edge $\{r', r''\}$. There are $2\bar{m} \equiv 2(k - k' - k'')$ such steps. If m' (resp. m'') is the number of returns of W' (resp. W'') at vertex r' (resp. r''), there are exactly $\binom{m' + \bar{m} - 1}{\bar{m} - 1}$ (resp. $\binom{m'' + \bar{m} - 1}{\bar{m} - 1}$) possibilities to insert the $\bar{m}$ steps from r' to r'' (resp. the $\bar{m}$ steps from r'' to r'). Note that the -1 in the above counting comes from the fact that the last (resp. first) step from r' to r''

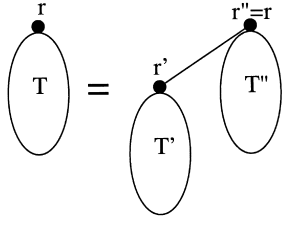

Fig. 6. Decomposition of T.

(resp. from r'' to r') is fixed. The total number of visits at vertex r'' is $\bar{m}+m''$, the first term counting visits from the edge $\{r', r''\}$ and the second visits from T''. If T' (resp. T'') has l' (resp. l'') edges, T has $l=l'+l''+1$ edges. To summarize, the number $\mathscr{I}_{k,l,m}$ of normalized $2k$-plets associated to trees with l edges and containing m times the number 1 satisfies the following recursion relation, for $k \geqslant 1$

$$\mathscr{I}_{k,l,m} = \sum \mathscr{I}_{k',l',m'} \mathscr{I}_{k'',l'',m''} \binom{m'+\bar{m}-1}{\bar{m}-1} \binom{m''+\bar{m}-1}{\bar{m}-1} \tag{8}$$

where the sums runs over non-negative indices k', k'', l', l'', m', m'' and $\bar{m}$ with relations $k'+k''+\bar{m}=k$, $l'+l''=l-1$ and $\bar{m}+m''=m$.

Note that $\mathscr{I}_{k,l,m}$ vanishes if $l>k$ (every edge is visited at least twice for a covering closed walk on a tree), if $m>k$ (two successive terms in a normalized $2k$-plet are distinct, so 1 cannot appear more than k times) and if $m=0$ unless $k=l=0$ (every non void normalized $2k$-plet contains 1). Finally, $\mathscr{I}_{0,0,0}=1$. This gives more than enough boundary conditions to compute the $\mathscr{I}_{k,l,m}$ recursively.

On a workstation, a day of symbolic computation with this formula gives the $\mathscr{I}_{k,l,m}$ as integers up to $k=50$. With 12 digits precision, a Fortran program goes to $k=120$ in about the same time. The results are available upon request to the authors.

Note that the index m is not directly relevant for the computation of moments, because $\mathscr{I}_{k,l}=\sum_m \mathscr{I}_{k,l,m}$, but we have not been able to obtain a closed recursion without the index m. Coefficients $\mathscr{I}_{k,l}$ for small k and l are given in Table I.

Table I. The Number of Normalized $2k$-Plets Associated to Trees with l Edges

$k\backslash l$	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	2								
3	1	6	5							
4	1	14	28	14						
5	1	30	110	120	42					
6	1	62	375	682	495	132				
7	1	126	1190	3248	3731	2002	429			
8	1	254	3628	14062	23020	18928	8008	1430		
9	1	510	10805	57516	127029	144024	91392	31824	4862	
10	1	1022	31740	227030	654395	968544	828495	426360	125970	16796

Let us note that each moment is a polynomial in α, so that in particular it does not exhibit any singularity at the percolation or localization transition. This is not very surprising, because by the general algorithm (any N and p) μ_k is computed by exploring connected subgraphs with at most k sites of the random graph. But as noted before a finite connected induced subgraph of a random graph at fixed α is a tree with probability 1. To see any trace of a transition, one would have to look at the behavior of μ_{2k} for large k. For instance, we tried to see if complex zeroes of μ_{2k} in the variable α (or other related quantities) have a tendency to accumulate near the real axis. We have found no conclusive evidence. The rapid growth of the coefficients $\mathscr{I}_{k,l}$ makes a numerical study up to $k=50$ difficult, even if we know all numbers exactly.

5.5. Bounds for $\mathscr{I}_{k,l}$

In this section, we will show that $\mathscr{I}_{k,l}$, the number of normalized $2k$-plets associated to trees with l edges satisfies the bounds

$$\mathscr{S}_{k,l} \leqslant \mathscr{I}_{k,l} \leqslant C_l \mathscr{S}_{k,l} \tag{9}$$

where C_l are Catalan numbers defined by

$$C_l = \frac{(2l)!}{l!\,(l+1)!}$$

and $\mathscr{S}_{k,l}$ are Stirling numbers of the second kind (i.e., the number of ways of partitioning a set of k elements into l non-empty subsets) which can be computed with the formula

$$\mathscr{S}_{k,l} = \frac{1}{l!} \sum_{m=0}^{l} (-1)^{l-m} \binom{l}{m} m^k. \tag{10}$$

Let W be a normalized $2k$-plet associated to a tree T. We view W as a walk on T. Then W allows to put more structure on T. First, the starting point of the walk turns T into a rooted tree. This allows to talk about sons of a vertex v, the root being the initial ancestor. Its sons are its neighbors, and so on. Then W also endows the sons of a vertex with an ordering: the order of first visit. This means that W naturally endows T with a structure of plane rooted tree, with the convention that the root is at the top of the tree, and the elder son is always the leftmost one. Then,

$$\mathscr{I}_{k,l} = \sum_{T_l} \mathscr{I}_k(T_l)$$

where the sum runs over the plane rooted trees T_l with l edges, and $\mathscr{I}_k(T_l)$ is the number of admissible walks on T_l with $2k$ steps, where *admissible* means starting and finishing at the root of T_l and visiting all the $l+1$ vertices by respecting the order of birth among brothers (i.e., a vertex can be visited only if its brothers on its left have been visited before).

First, we will prove that $\mathscr{I}_k(T_l^\star) = \mathscr{S}_{k,l}$ for a particular tree—the star-like tree—which gives the lower bound of Eq. (9). Then, we will prove that

$$\mathscr{I}_k(T_l) \leqslant \mathscr{S}_{k,l} \tag{11}$$

for any tree T_l. As the number of plane rooted trees with l edges is C_l (see, e.g., ref. 24), this gives the upper bound of Eq. (9).

The star-like tree $T_l^\star$ is made of a root r and l sons: $\{s_1, s_2, ..., s_l\}$, see Fig. 7. If $l=0$, then $\mathscr{I}_k(T_0^\star) = \delta_{k,0} = \mathscr{S}_{k,0}$. So we consider now that $l \geqslant 1$. All the normalized $2k$-plet associated to $T_l^\star$ can be written as $(r, s_{v(1)}, r, s_{v(2)}, ..., r, s_{v(k)})$ where v is an onto map from $[1,k]$ to $[1,l]$, because an admissible walk is made of k successive double-steps from r to a son and return, and visits all the sons. The inverse map v^{-1} makes a partition of $[1,k]$ into l non-empty subsets: $v^{-1}(j) = \{i \mid 1 \leqslant i \leqslant k;\ v(i) = j\}$ for $j \in [1,l]$. Conversely, such a partition gives a admissible walk, because the targets j associated to each subset are uniquely determined by the birth rule, with the following process: the subset containing 1 is labeled by $j=1$ and removed, the subset containing the smallest remaining number is labeled by $j=2$ and removed, etc, up to the last subset labeled by $j=l$. This is a one-to-one correspondence between the admissible walks and the partitions of $[1,k]$ into l non-empty subsets, which are counted by Stirling numbers of second kind. Hence, $\mathscr{I}_k(T_l^\star) = \mathscr{S}_{k,l}$ for the star-like tree $T_l^\star$.

Let us now consider a given plane rooted tree T_l with l edges. So Eq. (11) is true for $k < l$ because $\mathscr{I}_k(T_l) = 0$ (the walk is too short to visit all the edges), and for $k = l$ because $\mathscr{I}_l(T_l) = 1 = \mathscr{S}_{l,l}$ (an admissible walk on T_l of length $2l$ exists and is completely determined by the birth rule). Note that this implies that $\mathscr{I}_{l,l} = C_l$. For $k > l$, we will prove Eq. (11) by induction, assuming Eq. (11) to be true for every (k', l') when $k' < k$.

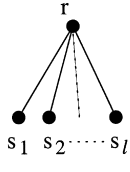

Fig. 7. Labels of vertices of the "star" tree $T_l^\star$.

Let r be the root of T_l and r' the leftmost son of r. As shown on Fig. 6, we break T_l in three parts, the edge $\{r, r'\}$, the sub-tree T' with root r' and l' edges and the rest T'', which is a sub-tree with root $r'' = r$ and l'' edges. We have $l = l' + l'' + 1$. An admissible walk W on T_l with $2k$ steps is composed of $2k'$ steps on T', $2k''$ steps on T'' and $2\bar{m}$ steps on the edge $\{r, r'\}$, with $k = k' + k'' + \bar{m}$ and $\bar{m} \geqslant 1$.

If T is a plane rooted tree, we call $\mathscr{I}_{k,m}(T)$ the number of admissible walks on T with $2k$ steps and m returns to its root, and define

$$\mathscr{H}_{k,\bar{m}}(T) \equiv \sum_{m=1}^{k} \mathscr{I}_{k,m}(T) \binom{m+\bar{m}-1}{\bar{m}-1}.$$

The arguments used to establish Eq. (8) can be repeated to show that

$$\mathscr{I}_k(T_l) = \sum_{\bar{m},k',k''}^{\bar{m}+k'+k''=k} \mathscr{H}_{k',\bar{m}}(T') \, \mathscr{H}_{k'',\bar{m}}(T''). \tag{12}$$

Now

$$\mathscr{H}_{k',\bar{m}}(T') \leqslant \sum_{m'=1}^{k'} \mathscr{I}_{k',m'}(T') \binom{k'+\bar{m}-1}{\bar{m}-1} = \mathscr{I}_{k'}(T') \binom{k'+\bar{m}-1}{\bar{m}-1}$$

and by the induction hypothesis, the last term is at most

$$\mathscr{I}_{k'}(T^{\star}_{l'}) \binom{k'+\bar{m}-1}{\bar{m}-1}.$$

But this is precisely $\mathscr{H}_{k',\bar{m}}(T^{\star}_{l'})$ because for each walk on $T^{\star}_{l'}$, $m' = k'$. Hence, $\mathscr{H}_{k',\bar{m}}(T') \leqslant \mathscr{H}_{k',\bar{m}}(T^{\star}_{l'})$. As the same argument holds for T'', $\mathscr{I}_k(T_l) \leqslant \mathscr{I}_k(T^{\star\star}_{l',l''})$ where $T^{\star\star}_{l',l''}$, is the "bi-star" tree with $T' = T^{\star}_{l'}$ and $T'' = T^{\star}_{l''}$.

It remains to show that $\mathscr{I}_k(T^{\star\star}_{l',l''}) \leqslant \mathscr{S}_{k,l'+l''+1}$. If $l' = 0$, it is true because the bi-star is simply the star $T^{\star}_{l}$. As Eq. (12) is symmetric by exchange between T' and T'', it is also true for $l'' = 0$ because $\mathscr{I}_k(T^{\star\star}_{l',0}) = \mathscr{I}_k(T^{\star\star}_{0,l'}) = \mathscr{S}_{k,l}$.

So we have just to consider the case $l' \geqslant 1$ and $l'' \geqslant 1$. The vertices of the bi-star are labeled as shown on Fig. 8. The admissible walks are divided in two classes: (I) the walks finishing by a step on the right subtree $T^{\star}_{l''}$ and (II) the walks finishing by a step from r' to r''.

For a walk W of class (I), we call $W^{\dagger}$ the walk W without the two last steps; $W^{\dagger}$ is made of $2k-2$ steps. The class (I) is divided in two subclasses: (I_1) $W^{\dagger}$ has not visited all the vertices. Then $W = (W^{\dagger}, r'', b_{l''})$ and $W^{\dagger}$ is an admissible walk on $T^{\star\star}_{l',l''-1}$. ($I_2$) $W^{\dagger}$ has visited all the vertices.

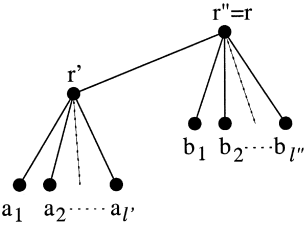

Fig. 8. Labels of vertices of the "bi-star" tree $T^{\star\star}_{l',l''}$.

Then $W^\dagger$ is an admissible walk on $T^{\star\star}_{l',l''}$ and there is l'' choices to built W from $W^\dagger$. In total, the number of walks of class (I) is $\mathscr{I}_{k-1}(T^{\star\star}_{l',l''-1}) + l''\mathscr{I}_{k-1}(T^{\star\star}_{l',l''})$, bounded above by $\mathscr{S}_{k-1,l-1} + l''\mathscr{S}_{k-1,l}$.

For a walk W of class (II), we call $W^\dagger$ the walk W with the first and last steps removed. Then $W^\dagger$ is made of $2k-2$ steps, starting and finishing at r' (and not r) and covering the bi-star. The delicate point is the moment of its first visit to r. We have $l'+1$ sub-classes. In sub-class (II_0), the first visit of r is before the first visit of a_1. In sub-class (II_i) with $1 \leqslant i \leqslant l'$, the first visit of r is after the first visit of a_i and before the first visit of a_{i+1}. In sub-class $(\mathrm{II}_{l'})$, the first visit of r is after the first visit of $a_{l'}$. By using the birth rule, each sub-class corresponds to the admissible walks on one of the plane rooted trees with l edges drawn on Fig. 9. For each of these $l'+1$ trees, the number of steps is $2k-2$, so that the induction hypothesis applies. Hence, for class (II), the total number of walks is bounded above by $(l'+1)\,\mathscr{S}_{k-1,l}$. Then, $\mathscr{I}_k(T^{\star\star}_{l',l''}) \leqslant \mathscr{S}_{k-1,l-1} + (l''+l'+1)\,\mathscr{S}_{k-1,l} = \mathscr{S}_{k-1,l-1} + l\mathscr{S}_{k-1,l}$. But, the Stirling numbers obey to the recursion relation $\mathscr{S}_{k,l} = \mathscr{S}_{k-1,l-1} + l\mathscr{S}_{k-1,l}$, so that we have reached our goal: Eq. (11) is proved and Eq. (9) follows.

We can now establish two important features of the eigenvalue distribution $d\rho_\alpha$: it's support is unbounded, but it Fourier–Laplace transform is an entire function, and in particular, the distribution of eigenvalues is determined by its moments.

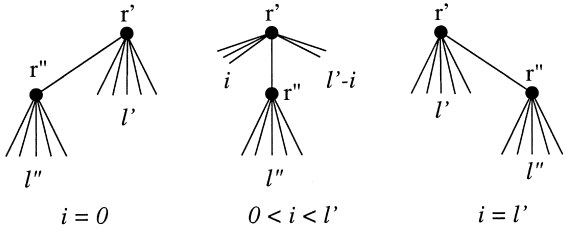

Fig. 9. The $l'+1$ plane rooted trees corresponding to the class (II).

By using the property of Stirling numbers $n^k = \sum_{l=0}^{n} \mathscr{S}_{k,l} n^{\underline{l}}$ and by summing it with $\sum x^n/n!$, one obtains

$$\mathscr{S}_k(x) \equiv \sum_{l=0}^{k} \mathscr{S}_{k,l} x^{\underline{l}} = e^{-x} \sum_{n=0}^{\infty} \frac{n^k}{n!} x^n.$$

This leads to a crude bound for the Stirling polynomials when x is real positive. In fact, $\mathscr{S}_0(x) = 1$, $\mathscr{S}_1(x) = x$, $\mathscr{S}_2(x) = x^2 + x$ and for $k \geqslant 3$, $\mathscr{S}_k(x) < k^k + e^{x(k-1)}$ because for $k \geqslant 3$ and any n, $n^k < k^k + k^n$.

To see the unboundedness of the support of the eigenvalue distribution, we observe that the Stirling polynomials $\mathscr{S}_k(\alpha)$ are the moments of even order of the even α-dependent probability measure $(e^{-\alpha}/2) \sum_{n=0}^{\infty} (\alpha^n/n!)$ $(\delta(x - \sqrt{n}) + \delta(x + \sqrt{n}))$ on the real line. The support of this measure is clearly unbounded. As the moments of $d\rho_\alpha$ are larger than the Stirling polynomials, the support of $d\rho_\alpha$ has to be unbounded.

To see the properties of the Fourier–Laplace transform, we just show that term by term expansion of $\int_{-\infty}^{+\infty} d\rho_\alpha(\lambda)\, e^{s\lambda}$ in powers of s leads to a series with infinite radius of convergence. For this, we just need to bound $\int_{-\infty}^{+\infty} d\rho_\alpha(\lambda)(\lambda^{2k}/(2k)!) = \mu_{2k}/(2k)!$. We know that this is at most $C_k \mathscr{S}_k(\alpha)/(2k)! = [1/(k!\,(k+1)!)]\, \mathscr{S}_k(\alpha)$. We have seen than for $k \geqslant 3$ this is less than $(k^k + e^{\alpha(k-1)})/(k!\,(k+1)!)$ which for large k behaves like $e^k/(\sqrt{2\pi k}\,(k+1)!) = o(R^{-k})$ for any R. So, in principle, the knowledge of the moments determines the probability distribution $d\rho_\alpha$. However, it is not easy to extract accurate local information on $d\rho_\alpha$ from the knowledge of a finite number of moments.

5.6. Special Values

Equation (8) can be used to compute $\mathscr{I}_{k,l}$ for special cases. This is best done using generating functions. Instead of giving the details, let us just note that this leads to closed forms for $\mathscr{I}_{k,l}$ for fixed l and any k or for fixed $k - l$ and any l, leading to a a satisfactory description of the borders of the table of moments. For $k = 0$, $\mathscr{I}_{0,l} = \delta_{0,l}$. For $k \geqslant 1$, the first cases are:

$$\mathscr{I}_{k,0} = 0$$

$$\mathscr{I}_{k,1} = 1$$

$$\mathscr{I}_{k,2} = 2^k - 2$$

$$\mathscr{I}_{k,3} = 3^{k-1} + \omega^k + \bar{\omega}^k - 3 \cdot 2^k + 2$$

with $\omega + \bar{\omega} = 3$ and $\omega\bar{\omega} = 1$.

On the other border

$$\mathscr{I}_{l,l} = \frac{(2l)!}{l!\,(l+1)!} = C_l,$$

$$\mathscr{I}_{l+1,l} = \frac{(2l+2)!}{(l-1)!\,(l+3)!},$$

$$\mathscr{I}_{l+2,l} = \frac{(2l+4)!}{(l-1)!\,(l+6)!}\,\frac{l^2+11l+2}{2},$$

$$\mathscr{I}_{l+3,l} = \frac{(2l+6)!}{(l-1)!\,(l+9)!}\,\frac{l^4+32l^3+323l^2+232l-48}{6}.$$

That $\mathscr{I}_{l,l}$ is just the Catalan numbers is not surprising for two reasons. First combinatorially, for any plane rooted tree with l edges, $\mathscr{I}_l(T_l)=1$, so $\mathscr{I}_{l,l}=C_l$, the number of plane rooted trees. Second, we know that for fixed p, the distribution of eigenvalues is governed by the semicircle law. It is not surprising that when α goes to infinity, the same distribution reappears. Indeed, the fact that $\mathscr{I}_{l,l}=C_l$ is equivalent to the fact that the typical size of eigenvalues is $\sqrt{\alpha}$, and that after the rescaling $\lambda=\sqrt{\alpha}\,x$, $d\rho_\alpha(\lambda)$ converges to semicircle law $1/2\pi\,\sqrt{4-x^2}\,dx$, for which the even moments are the Catalan numbers.

The above equations are reminiscent of the meanders problem.[8] In particular, we observe that diagonals of Table I verify

$$\mathscr{I}_{l+u,l} = \frac{(2l+2u)!}{(l-1)!\,(l+3u)!}\,\frac{P_{2u-2}(l)}{u!}$$

where $P_{2u-2}(l)$ is a polynomial with integer coefficients and with leading term l^{2u-2}. By conjecturing this form, the coefficients could be determined for the first values of u, from the exact knowledge of the first $\mathscr{I}_{l+u,l}$. Unfortunately, we know no general formula for $\mathscr{I}_{k,l}$.

The asymptotics of $\mathscr{I}_{k,l}$ for fixed $l\geqslant 2$ and large k are governed by a simple relation,

$$\mathscr{I}_{k,l} \sim 2\mathscr{S}_{k,l} \sim 2\,\frac{l^k}{l!}. \tag{13}$$

As explained in Section 5.5, $\mathscr{I}_{k,l}=\sum_{T_l}\mathscr{I}_k(T_l)$, and $\mathscr{I}_k(T_l)$ is maximal for the star tree $T_l^\star$ as for the bi-star tree $T^{\star\star}_{l-1,0}$ (isomorphic to the star but with a leaf as root), with $\mathscr{I}_k(T_l^\star)=\mathscr{I}_k(T^{\star\star}_{l-1,0})=\mathscr{S}_{k,l}$. Hence, $\mathscr{I}_{k,l}\geqslant 2\mathscr{S}_{k,l}$ for $l\geqslant 2$. But, for l fixed, we will show that $\mathscr{I}_{k,l}$ is asymptotic

to this lower bound when k is large, because among all the C_l plane rooted trees, the contributions of $T_l^{\star}$ and $T_{l-1,0}^{\star\star}$ become dominant. Equation (10) says that $\mathscr{S}_{k,l} \sim l^k/l!$ for large k.

Let M be the incidence matrix of a given plane rooted tree T_l with l edges: the number of admissible walks with $2k$ steps, $\mathscr{I}_k(T_l)$, is bounded above by $\mathrm{Tr}\, M^{2k}$ which counts the closed walks on T with $2k$ steps. So T_l contributes for large k only if the Perron–Frobenius eigenvalue of M is not smaller than $\sqrt{l}$.

A tree can be bicolored: if we use black (B) and white (W) as the colors, each vertex is either black or white, in such a way that edges connect only vertices with opposite colors. Then, for an eigenvector ϕ with eigenvalue λ, the vector ϕ', defined by $\phi'_i = \phi_i$ on black vertices and $-\phi_i$ on white vertices, is eigenvector associated to eigenvalue $-\lambda$. Hence, the spectrum is symmetric.

As T_l has l edges, $\mathrm{Tr}\, M^2 = \sum_{i,j} M_{i,j}^2 = 2l$, and for the spectrum of M, $\sum_i \lambda_i^2 = 2l$. We see that T_l contribute only if its spectrum is $\{-\sqrt{l}, 0, \sqrt{l}\}$ where 0 is $l-2$ times degenerated. By noting Φ the Perron–Frobenius eigenvector, associated to $\sqrt{l}$, and Φ' the eigenvector associated to $-\sqrt{l}$, in this case

$$M_{i,j} = \sqrt{l}\,(\Phi_i \Phi_j - \Phi'_i \Phi'_j).$$

As Φ' is obtained by inverting the signs of Φ on white vertices, $M_{i,j} = 0$ when i and j have the same color and $M_{i,j} = 2\sqrt{l}\,\Phi_i \Phi_j$ when i and j have different colors. The Perron theorem assures that $\Phi_i > 0$ for all i. Hence, $M_{i,j} = 1$ when i and j have different colors. As loops are forbidden in a tree, one of the colors must color only one vertex. So M must be the incidence matrix of a tree isomorphic to the star $T_l^{\star}$, which prove Eq. (13).

Using the saddle point approximation, it is not difficult to show that for large k and l with k/l fixed, $\mathscr{S}_{k,l}$ grows faster than any exponential of l. In this regime, $\log \mathscr{I}_{k,l} \sim \log \mathscr{S}_{k,l}$ because $\log C_l \sim l \log 4 \ll \log \mathscr{S}_{k,l}$. But the asymptotics of $\mathscr{I}_{k,l}$ seem much more complicated.

6. CONCLUSION

In this paper we have made a detailed study of the spectral density of large random graphs incidence matrices, both in the fixed edge probability p and in the fixed average connectivity α limits.

For fixed p, our results are:

- A simple general algorithm to compute arbitrary moments (polynomials in N and p), easy to implement on a computer (but of almost factorial growth), leading to an explicit form for the first 18 moments.

• Numerical diagonalizations of Monte Carlo samples of random matrices and analytical arguments to show that when the random signs are not chosen with a symmetric probability, the main change is that a "Perron Frobenius" eigenvalue has to be eliminated to recover the semicircle law and the GOE results.

For fixed α (the finite connectivity limit), our main contributions are:

• Numerical diagonalizations of Monte Carlo samples of random matrices for different values of α, with a curious observation for $\alpha \simeq 2.7$.

• A general proof that the spectrum contains delta peaks at all eigenvalues of tree incidence matrices, and that, slightly surprisingly, they receive non-vanishing contributions even from the infinite cluster when $\alpha > 1$.

• A recursion relation of combinatorial origin to compute the moments. Odd moments vanish, and we have explicitly computed the first 120 even moments with the help of a computer. Our algorithm counts objects which are related to many other tree enumeration problems of independent interest.

• A proof that the growth of the moment is slow enough so that they determine the spectral distribution entirely.

• The moments are insensitive to the random signs, and so is the spectrum by the previous remark.

• The moments are polynomials in α. In particular, they do not exhibit any singularity at the classical or quantum percolation transitions.

• However, we have given a general formula (7) to prepare the ground for a more refined study[3] of the delta peak at the eigenvalue $\lambda = 0$ in the spectral distribution. This delta peak is directly relevant to quantum percolation and is non-analytic in α at $\alpha = e$, which we believe is connected to the curious numerical observation at $\alpha \simeq 2.7$ mentioned above.

One of the natural continuations of this work would be a careful numerical and/or analytical study of moments of large order, to see if the percolation transitions have a measurable impact on global characteristics of the spectral density.

From a more mathematical point of view, and because of their close connection with delta peaks in the spectra of random graphs, it might be of interest to be able to characterize the eigenvalues of tree incidence matrices of a given size and how their distribution and spacings evolve for large sizes.

APPENDIX. RANDOM LAPLACIAN MATRIX

In this appendix, we give results for the Laplacian matrix on random graphs. These are obtained using an adaptation of methods previously described.

The random Laplacian matrix L of size N is defined as $L = D - M$ where M is a random incidence matrix (see Section 2) with vanishing diagonal elements and D is the diagonal matrix whose element $D_{ii} = \sum_{j \neq i} M_{ij}$ is the connectivity of the vertex i in the random graph associated to M. So all rows or column sums of L vanish.

The Perron–Frobenius theorem applies to $N - L$: the Perron eigenvector is the uniform eigenvector $(1,..., 1)^T$, associated to the eigenvalue $\lambda_P = 0$ for L. It follows that the spectrum of L is real and non-negative. Moreover the multiplicity of 0 is the number of connected components.

To compute $\overline{\mathrm{Tr}\, L^k}$, the method explained in Section 3.1 is adapted to the Laplacian. After expanding $(D - M)^k$, and eventually using the invariance of the trace by cyclic permutation, we must compute several terms individually. Of course, the diagonal elements D_{ii} need a special treatment. With the trick $D_{ii} = \sum_{j \neq i} M_{ij} = \sum_{j \neq i} M_{ij} M_{ji}$, D_{ii} becomes a double-step (i, j, i) in the description as a walk on a graph. Now, an admissible k-plet is made up of single-steps (for M) and double-steps (for D). After enumeration,

$$\overline{\mathrm{Tr}\, L} = pN^2,$$

$$\overline{\mathrm{Tr}\, L^2} = p^2 N^3 + 2pN^2,$$

$$\overline{\mathrm{Tr}\, L^3} = p^3 N^4 + (6p^2 - p^3)\, N^3 + 4pN^2,$$

$$\overline{\mathrm{Tr}\, L^4} = p^4 N^5 + (12p^3 - 3p^4)\, N^4 + (25p^2 - 6p^3)\, N^3 + 8pN^2.$$

A computer program can expand these results for a dozen of k. They can be checked with sum rules: for $p = 1$ the matrix is deterministic and $\overline{\mathrm{Tr}\, L^k} = (N-1)\, N^k$.

For finite N there are in general several connected components, hence the Perron eigenvalue $\lambda_P = 0$ appears with multiplicity. However for fixed p the random graph is connected in the large N limit, and $\lambda_P = 0$ appears only once. So it is reasonable to consider centered moments $m_k = \overline{(\lambda - \bar{\lambda})^k}$ where the means run on the other $N - 1$ eigenvalues. From previous equations,

$$m_2 = 2p(1-p)\, N,$$

$$m_3 = 4p(1-p)(1-2p)\, N,$$

$$m_4 = p(1-p)[\, p(1-p)(9N-42) + 8\,]\, N.$$

In the large N limit with p fixed, $m_3/m_2^{3/2}$ goes to 0 and m_4/m_2^2 goes to 9/4. The limit distribution has a bell-like shape,[5, 7] intermediate between the semicircle and the Gauss law for which m_4/m_2^2 is 2 and 3 respectively.

In the large N limit with $\alpha = pN$ fixed, the k-plets contributing to the dominant term of order N are associated to trees, as described in Section 5.3. So the kth moment is a polynomial in α of degree k,

$$\mu_k \equiv \lim_{N\to\infty} \frac{1}{N} \overline{\mathrm{Tr}\, L^k} = \sum_l \mathscr{L}_{k,l}\alpha^l$$

where $\mathscr{L}_{k,l}$ is the number of normalized k-plets (with single or double steps) associated to trees with l edges. Note that now k is not constrained to be even, due to the double steps. Following the arguments and notations of Section 5.4, we call $\mathscr{L}_{k,l,m}$ the number of normalized k-plets associated to trees with l edges and containing m times the number 1 (i.e., with m return to the root of the associated tree).

In comparison with Section 5.4, as the steps on the first edge $\{r', r''\}$ are single or double, we call u' (resp. u'') the number of such steps finishing on r' (resp. r''), i.e., single steps (r'', r') and double steps (r', r'', r') (resp. (r', r'') and (r'', r', r'')). Then for $k \geqslant 1$, $\mathscr{L}_{k,l,m}$ satisfies the recursion relation

$$\mathscr{L}_{k,l,m} = \sum \mathscr{L}_{k',l',m'}\mathscr{L}_{k'',l'',m''} \binom{m'+u'-1}{u'-1}\binom{m''+u''-1}{u''-1}\binom{u'+u''-1}{u'}$$

where the sum runs over non-negative indices k', k'', l', l'', m', m'', u' and u'' with relations $k'+k''+u'+u''=k$, $l'+l''=l-1$ and $u''+m''=m$, with

Table II. The Number of Normalized 2k-Plets, with Single or Double Steps, Associated to Trees with l Edges

$k\backslash l$	1	2	3	4	5	6	7	8	9	10
1	1									
2	2	1								
3	4	6	1							
4	8	25	12	1						
5	16	90	85	20	1					
6	32	301	476	215	30	1				
7	64	966	2345	1722	455	42	1			
8	128	3025	10696	11659	4928	854	56	1		
9	256	9330	46453	71082	43779	12012	1470	72	1	
10	512	28501	195340	404540	342642	135357	26040	2370	90	1

Table III. Coefficients $\mathscr{L}_{k,l}^{(c)}$ of the Centered Moments of the Random Laplacian Spectral Distribution

$k\backslash l$	1	2	3	4	5	6
2	2					
3	4					
4	8	9				
5	16	50				
6	32	205	56			
7	64	742	574			
8	128	2513	3864	431		
9	256	8178	21532	6906		
10	512	25941	107800	68455	3942	
11	1024	80894	504394	540782	90508	
12	2048	249337	2255128	3739054	1240360	42136

the convention that the first binomial coefficient must be taken as 1 (and not 0) when $m' = u' = 0$. To start the recursion relation, we need the boundary conditions for $k = 0$: $\mathscr{L}_{0,l,m} = \delta_l \delta_m$.

By summing on m, we retrieve the coefficients $\mathscr{L}_{k,l} = \sum_m \mathscr{L}_{k,l,m}$. The first of these are given in Table II. As the average $\mu_1 = \alpha$ is not zero, we compute the *centered* moments m_k, which are still polynomial in α of degree the integer part of $k/2$. The coefficients $\mathscr{L}_{k,l}^{(c)}$ are given in Table III. The shape of the distribution evolves with α: in particular, it becomes non-symmetric as the odd centered moments do not vanish. Again, we observe that the results for large N with fixed p are found by keeping the last diagonal of Tables II and III, corresponding to the large α limit.

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